General Weak Limit for Kähler-Ricci Flow

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1 Introduction

X is a closed Kähler manifold with $\dim_{\mathbb{C}} X = n \geqslant 2$. We consider the following Kähler-Ricci flow over X,

$$\frac{\partial \widetilde{\omega}(t)}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}(t)) - \widetilde{\omega}(t), \qquad \widetilde{\omega}(0) = \omega_0. \tag{1.1}$$

Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ with $\omega_\infty = -\mathrm{Ric}(\Omega)$. It's known that $\widetilde{\omega}(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ with u satisfying

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \qquad u(\cdot, 0) = 0. \tag{1.2}$$

By the optimal existence result for general Kähler-Ricci flow in [7], the smooth solution exists as long as $[\omega_t]$ stays in the Kähler cone of X in the cohomology space $H^2(X;\mathbb{R}) \cap H^{1,1}(X;\mathbb{C})$. We denote the time of singularity by T, where $[\omega_T]$ is on the boundary of the Kähler cone for X. The flow exists for $[0, T)$ for some $0 < T \leqslant \infty$.

In general, as shown in Tian's survey [6], one has a sequential limit as $t \to T$ in the weak (i.e. current) sense, noticing that the argument there works for $T = \infty$ for our version of Kähler-Ricci flow because ω_t is uniformly controlled as form even when $T = \infty$. Normalization of the metric potential u is performed to achieve the limit, and it's also conjectured there that the sequential limit is unique. We attack this topic by studying u itself along the flow without normalization. If the flow limit exists, then the sequential limit is unique.

The standard simplification of notation is applied with C standing for a positive constant, possibly different at places. Also, $f \sim g$ means $\lim_{t\to T} \frac{f}{g} = 1$.

2 Known cases

We already have the following computation as in [7]. The Laplacian, Δ, in this note is always with respect to the metric along the flow, $\widetilde{\omega}_t$.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle \widetilde{\omega}_t, \omega_0 - \omega_\infty \rangle - \frac{\partial u}{\partial t}, \tag{2.1}$$

which is just the t-derivative of (1.2). It can be transformed into the following two equations,

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right)=\Delta\left(e^t\frac{\partial u}{\partial t}\right)-\langle\widetilde{\omega}_t,\omega_0-\omega_\infty\rangle,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right)=\Delta\left(\frac{\partial u}{\partial t}+u\right)-n+\langle\widetilde{\omega}_t,\omega_\infty\rangle.$$

The difference of these two equations gives

$$\frac{\partial}{\partial t}\left((1-e^t)\frac{\partial u}{\partial t}+u\right)=\Delta\left((1-e^t)\frac{\partial u}{\partial t}+u\right)-n+\langle\widetilde{\omega}_t,\omega_0\rangle.$$

By Maximum Principle, this implies the essential decreasing (i.e. up to an exponentially decaying term) of the metric potential along the flow as follows,

$$\frac{\partial u}{\partial t}\leqslant\frac{nt+C}{e^t-1}.$$

Notice that this estimate only depends on the initial value of u and its upper bound along the flow.

Another t-derivative for (2.1) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right)=\Delta\left(\frac{\partial^2 u}{\partial t^2}\right)+e^{-t}\langle\widetilde{\omega}_t,\omega_0-\omega_\infty\rangle-\frac{\partial^2 u}{\partial t^2}-\left|\frac{\partial\widetilde{\omega}_t}{\partial t}\right|^2_{\widetilde{\omega}_t}. \tag{2.2}$$

Take summation with (1.2) to get

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)=\Delta\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)-\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)-\left|\frac{\partial\widetilde{\omega}_t}{\partial t}\right|^2_{\widetilde{\omega}_t}.$$

Maximum Principle then gives

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\leqslant Ce^{-t},$$

which implies the essential decreasing of volume form along the flow, i. e.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right)\leqslant Ce^{-t}.$$

From this, it's also easy to see

$$\frac{\partial u}{\partial t}\leqslant Ce^{-t}.$$

Hence, we know that $u-Ce^{-t}$ is decreasing along the flow. As long as this term (or equivalently u) doesn't converge to $-\infty$ uniformly when approaching the singular time T, which can be ∞ at this moment, u would converge to $u_T\in PSH_{\omega_T}(X)$ and $\omega_T+\sqrt{-1}\partial\bar{\partial}u_T$ is the weak flow limit for $\widetilde{\omega}_t$ as $t\to T$. This is true for the cases studied in [7].

2.1 Cases from algebraic geometry background

If $T < \infty$ and $[\omega_T]$ is semi-ample (or slightly more generally, there exists a smooth non-negative representative of this class), then the limit is unique. This is because we actually know $u \geqslant -C$ in this case by direct flow argument using Maximum Principle. This is already done in [7] and we include the detail for later convenience.

For any $S < \infty$, we have

$$\frac{\partial}{\partial t}\left((1 - e^{t-S}\frac{\partial u}{\partial t} + u\right) = \Delta\left((1 - e^{t-S})\frac{\partial u}{\partial t} + u\right) - n + \langle \widetilde{\omega}_t, \omega_S \rangle$$

by a proper linear combination of the two transformed equations of (2.1). Then let's choose S to be the time T of singularity, assumed to be finite here.

The setting at the beginning gives $f \in C^\infty(X)$ with $\omega_T + \sqrt{-1}\partial\bar{\partial} f \geqslant 0$. Modify the above equation a little bit as follows

$$\frac{\partial}{\partial t}\left((1 - e^{t-T}\frac{\partial u}{\partial t} + u - f\right) = \Delta\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u - f\right) - n + \langle \widetilde{\omega}_t, \omega_T + \sqrt{-1}\partial\bar{\partial} f \rangle.$$

Applying Maximum Principle and $T < \infty$, one has

$$(1 - e^{t-T})\frac{\partial u}{\partial t} + u - f \geqslant -C.$$

As $u \leqslant C$ and $\frac{\partial u}{\partial t} \leqslant C$, we can conclude that

$$u \geqslant -C, \quad \frac{\partial u}{\partial t} \geqslant -\frac{C}{1 - e^{t-T}} \sim -\frac{C}{T - t}.$$

In fact, we can get the lower bound for u more directly by looking at (1.2). After a proper choice of Ω, we can make sure $\omega_T \geqslant 0$, and so $\omega_t \geqslant C(T - t)\omega_0$ for $t \in [0, T)$. Then by Maximum Principle, we have

$$\frac{d \min_{X \times \{t\}} u}{dt} \geqslant n \log(T - t) - C - \min_{X \times \{t\}} u.$$

One can easily get the lower bound of u from this. This is also observed in [4].

Together with the essential decreasing of u known in general, we have the limit of u as a flow limit for $t \to T$ in $PSH_{\omega_T}(X) \cap L^\infty(X)$.

Hence in this case, one has the flow weak convergence for all wedge powers of $\widetilde{\omega}_t$, $\widetilde{\omega}_t^k$ for $k = 1, \cdots, n$.

Remark 2.1. *It remains an interesting question to see whether is the limit of u is continuous, especially for the collapsing case, i.e. when $[\omega_T]^n = 0$.*

The weak convergence result can be easily generalized to the case when $[\omega_T] - D$ has a non-negative representative, where D is an effective $\mathbb{R}$-divisor. The previous argument can be carried through with minor changes.

For simplicity, we assume D is an effect $\mathbb{Z}$-divisor and the generalization is trivial. D can be seen as a holomorphic line bundle which has a defining section $\{\sigma = 0\}$ and a hermitian metric $|\cdot|$. We can get this information involved in the equation applied before as follows.

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t} + u - \log|\sigma|^2\right) = \Delta\left((1-e^{t-T})\frac{\partial u}{\partial t} + u - \log|\sigma|^2\right) - n \\ + \langle \widetilde{\omega}_t, \omega_T + \sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle$$

Since $[\omega_T] - D$ has a non-negative representative, by choosing $|\cdot|$ properly, we have

$$\omega_T + \sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 \geqslant 0.$$

So by Maximum Principle, one has

$$(1-e^{t-T})\frac{\partial u}{\partial t} + u - \log|\sigma|^2 \geqslant -C.$$

As $u \leqslant C$ and $\frac{\partial u}{\partial t} \leqslant C$, we can conclude that

$$u \geqslant \log|\sigma|^2 - C, \quad \frac{\partial u}{\partial t} \geqslant \frac{C\log|\sigma|^2}{1-e^{t-T}} \sim \frac{C\log|\sigma|^2}{T-t}.$$

This (degenerate) lower bound for u also guarantees the flow limit $u_T \in PSH_{\omega_T}$ and the weak convergence of $\widetilde{\omega}_t$ to $\omega_T + \sqrt{-1}\partial\bar{\partial}u_T$ as $t \to T$.

Remark 2.2. *The case when $[\omega_T] = D$, an effective divisor, is included in the above case.*

2.2 Global volume non-collapsing case

In this part, we show that if $[\omega_T]^n > 0$, then u can not go to $-\infty$ uniformly, and so $u \to u_T \in PSH_{\omega_T}(X)$. Obviously, it is always the case that $[\omega_T]^n \geqslant 0$ since it is the limit of $[\omega_t]^n > 0$ as $t \to T$. Here we exclude the case of $[\omega_T]^n = 0$, i.e. require the flow to be (globally) volume non-collapsing.

The proof is very simple. We just need to make one observation after rewriting (1.2) as follows:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t}+u}\Omega. \tag{2.3}$$

The argument then could go by contradiction. Assuming otherwise, the decreasing limit of u would have to be $-\infty$ over X, i.e. it converges to $-\infty$ uniformly over X. Meanwhile, $[\omega_t]^n] \geqslant C > 0$ for t close to T since $[\omega_t]^n \to [\omega_T]^n > 0$ as $t \to T$. Also, $\frac{\partial u}{\partial t} \leqslant C$ in general. In sight of

$$\int_X e^u\Omega = C\int_X (\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = C[\omega_t]^n \geqslant C > 0,$$

we arrive at a contradiction. Notice that this argument works for $T \in (0, \infty]$.

Remark 2.3. *In general, one has* $[\omega_t]^n \sim C(T-t)^k$ *for some* $k \in \{0, \cdots, n\}$ *when* $T < \infty$ *and* $[\omega_t]^n \sim Ce^{-kt}$ *for some* $k \in \{0, \cdots, n\}$ *when* $T = \infty$. *This can be seen by analyzing the Taylor series of the fairly explicit function* $f(t) = [\omega_t]^n$ *at* $t = T$.

Now let's briefly the metric extimate in [8]. Begin with the inequality from parabolic Schwarz Lemma. Let $\phi = \langle \widetilde{\omega}_t, \omega_0 \rangle > 0$. Using computation for (1.1) in [3], one has

$$(\frac{\partial}{\partial t} - \Delta)\log\phi \leqslant C_1\phi + 1, \tag{2.4}$$

where C_1 is a positive constant depending on the bisectional curvature of ω_0. Also Recall the equation

$$\frac{\partial}{\partial t}\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right) = \Delta\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right) - \langle \widetilde{\omega}_t, \omega_0 \rangle. \tag{2.5}$$

and so $(e^t-1)\frac{\partial u}{\partial t} - u - nt \leqslant C$.

Multiplying (2.5) by a large enough constant $C_2 > C_1 + 1$ and combining it with (2.4), one arrives at

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\left(\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right)\right) &\leqslant nC_2 + 1 - (C_2 - C_1)\phi \\ &\leqslant C_3 - \phi.\end{aligned} \tag{2.6}$$

Now apply Maximum Principle for the term $\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right)$. Considering the place where it achieves maximum value, one has

$$\phi \leqslant C,$$

and so

$$\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right) \leqslant C,$$

which gives

$$\widetilde{\omega}_t \leqslant Ce^{-C_2\left((e^t-1)\frac{\partial u}{\partial t} - u - nt\right)}\omega_0 \leqslant Ce^{-C(e^t\frac{\partial u}{\partial t} - t)}\omega_0.$$

Since $\widetilde{\omega}_t^n = e^{\frac{\partial u}{\partial t} + u}\Omega$, we can further conclude that

$$Ce^{C(e^t\frac{\partial u}{\partial t} - t)}\omega_0 \leqslant \widetilde{\omega}_t \leqslant Ce^{-C(e^t\frac{\partial u}{\partial t} - t)}\omega_0.$$

Now we restrict to the finite time singularity case and have

$$Ce^{C\frac{\partial u}{\partial t}}\omega_0 \leqslant \widetilde{\omega}_t \leqslant Ce^{-C\frac{\partial u}{\partial t}}\omega_0 \quad \text{for} \quad t \in [0, T).$$

So the control of metric itself is totally reduced to the lower bound of $\frac{\partial u}{\partial t}$. Although we know from [8] that it's impossible to have a uniform lower bound, but this pointwise control is local and can still be helpful.

2.3 Some geometric cases

In this part, we discuss a couple of cases with stadard curvature assumptions.

Case 1: uniform Ricci lower bound for finite time singularity

In other words, we have $\mathrm{Ric}(\widetilde{\omega}_t) \geqslant -C\cdot\widetilde{\omega}_t$ for $t \in [0,T)$, and $T < \infty$. Clearly, $\widetilde{\omega}_t \leqslant C\omega_0$ from the flow equation itself. Thus we have

$$-e^{-t}(\omega_0 - \omega_\infty) + \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t} = \frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t \leqslant C\widetilde{\omega}_t \leqslant C\omega_0,$$

which gives

$$C\omega_0 + \sqrt{-1}\partial\bar{\partial}\left(-\frac{\partial u}{\partial t}\right) \geqslant 0.$$

By the result in [5], we know there exist uniform constants $0 < \alpha < 1$ and $C > 0$ such that

$$\int_X e^{\alpha\left(\max_X(-\frac{\partial u}{\partial t}) - (-\frac{\partial u}{\partial t})\right)}\Omega \leqslant C, \text{ uniformly for any } t \in [0,T).$$

That is

$$\int_X e^{\alpha\left((-\min_X \frac{\partial u}{\partial t}) + \frac{\partial u}{\partial t}\right)}\Omega \leqslant C.$$

So we have

$$\int_X e^{\alpha\frac{\partial u}{\partial t}}\Omega \leqslant Ce^{\alpha \min_X \frac{\partial u}{\partial t}} \leqslant Ce^{C\int_X \frac{\partial u}{\partial t}\Omega}. \tag{2.7}$$

Meanwhile, by Remark 2.3, we know

$$\int_X e^{\frac{\partial u}{\partial t}+u}\Omega = [\omega_t]^n \geqslant C(T-t)^k$$

for some $k \in \{0, \cdots, n\}$. Together with $\alpha < 1$ and the uniform upper bounds for u and $\frac{\partial u}{\partial t}$, it says

$$\int_X e^{\alpha\frac{\partial u}{\partial t}}\Omega \geqslant C\int_X e^{\frac{\partial u}{\partial t}+u}\Omega \geqslant C(T-t)^k. \tag{2.8}$$

Combining (2.7) with (2.8), we have

$$\int_X \frac{\partial u}{\partial t}\Omega \geqslant C\log(T-t) - C$$

which implies $\int_X u\Omega \geqslant -C$. This would be enough to conclude the limit of u in $PSH_{\omega_T}(X)$. We can actually do much better than that. Notice that (2.7) can be modifed to

$$\int_X e^{\alpha\frac{\partial u}{\partial t}}\Omega \leqslant Ce^{\alpha\min_X\left(\frac{\partial u}{\partial t}\right)} \leqslant Ce^{\alpha\frac{\partial u}{\partial t}(x_{\min}(t),t)} \tag{2.9}$$

where $x_{\min}(t)$ is a point where $u(\cdot, t)$ takes the minimum. Define the Lipschitz function $U(t) = \min_{X\times t} u$ and we have $\frac{dU}{dt} = \frac{\partial u}{\partial t}(x_{\min}(t), t)$. Now combining (2.9) with (2.8), we have

$$\frac{dU}{dt} \geqslant C \log(T - t) - C$$

which gives $U \geqslant -C$. So we actually have a uniform L^∞-bound for u. We summarize this in the following proposition.

Proposition 2.4. *For the Kähler-Ricci flow (1.1), if there is finite time singularity while Ricci curvature has uniform lower bound, then the metric potential in (1.2) has uniform L^∞-bounded. Along the flow, the metric weakly converges to a positive $(1,1)$-current with bounded potential, together with all the corresponding wedge powers*

Case 2: Type I singularity.

This is finite time singularity case, and we only need $R \leqslant \frac{C}{T-t}$. In sight of the volume evolution

$$\frac{\partial \widetilde{\omega}_t^n}{\partial t} = (-R - n)\widetilde{\omega}_t^n,$$

we have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \geqslant -\frac{C}{T-t}.$$

This tells us $\frac{\partial u}{\partial t} + u \geqslant C \log(T - t) - C$. Again, we can conclude that

$$u \geqslant -C.$$

Thus the weak flow convergence is clear.

In summary, for these two geometric cases of fairly common interest, the (lower) bound of the metric potential can be viewed as a test for the curvature conditions along the flow.

3 Remaining cases

Recall that for the uniqueness of sequential limit (or the existence of flow limit) to be true in general, we need to rule out the scenario of $u \to -\infty$ uniformly over X as $t \to T$. Since the volume non-collapsing case is dealt with for $T \in (0, \infty]$ in Section 2, we shall focus on the case when $[\omega_T]^n = 0$, and so $[\omega_t]^n \sim C(T - t)^k$ or Ce^{-kt} for some $k \in \{1, \cdots, n\}$, depending on whether $T < \infty$ or $T = \infty$).

For the rest of this section, we derive related estimates and discuss strategy to analyze the possibility of $u \to -\infty$ uniformly over X.

3.1 Estimates for collapsing case

In the collapsing case, we have

$$\int_X e^{\frac{\partial u}{\partial t}+u}\Omega = [\omega_t]^n \to 0 \text{ as } t \to T.$$

Using the estimate, $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right) \leqslant Ce^{-t}$ in Section 2, we conclude

$$\frac{\partial u}{\partial t}+u \to -\infty \text{ uniformly as } t \to T.$$

Then the following equation

$$\frac{\partial}{\partial t}\left((e^t-1)\frac{\partial u}{\partial t}-u\right) = \Delta\left((e^t-1)\frac{\partial u}{\partial t}-u\right)+n-\langle\widetilde{\omega}_t,\omega_0\rangle,$$

together with

$$\langle\widetilde{\omega}_t,\omega_0\rangle \geqslant n\left(\frac{\omega_0^n}{\widetilde{\omega}_t^n}\right)^{\frac{1}{n}} \geqslant Ce^{-\frac{1}{n}\left(\frac{\partial u}{\partial t}+u\right)} \to +\infty \text{ as } t \to T.$$

So we have

$$(e^t-1)\frac{\partial u}{\partial t}-u \leqslant -At+C(A)$$

for any (large positive) constant A with $C(A)$ depending on A. This won't say much for $T<\infty$, but we actually know this term also tends to $-\infty$ uniformly as follows. Recall another equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right) = \Delta\left(\frac{\partial u}{\partial t}+u\right)-n+\langle\widetilde{\omega}_t,\omega_\infty\rangle.$$

Considering t close to T, we have

$$n-\langle\widetilde{\omega}_t,\omega_0\rangle \leqslant -n-(1-\delta)\langle\widetilde{\omega}_t,\omega_0\rangle \leqslant -\epsilon n+\epsilon\langle\widetilde{\omega}_t,\omega_\infty\rangle$$

for some positive constants δ and ϵ. So Maximum Principle applying to the evolution of

$$\epsilon\left(\frac{\partial u}{\partial t}+u\right)-\left((e^t-1)\frac{\partial u}{\partial t}-u\right)$$

would tell us

$$(e^t-1)\frac{\partial u}{\partial t}-u \leqslant \epsilon\left(\frac{\partial u}{\partial t}+u\right)+C,$$

and so

$$(e^t-1)\frac{\partial u}{\partial t}-u \to -\infty \text{ uniformly as } t \to T.$$

Clearly $(e^t-1)\frac{\partial u}{\partial t}$ also tends to $-\infty$ uniformly as $t \to T$.

Now let's restrict ourselves to the case of $T<\infty$, i.e. finite time singularity.

First, we now immediately see

$$\frac{\partial u}{\partial t} \to -\infty \text{ uniformly as } t \to T.$$

Also from the above control of $(e^t-1)\frac{\partial u}{\partial t} - u$, we have

$$\frac{\partial u}{\partial t} \leqslant Cu - B(t)$$

for some $B(t) \to \infty$ as $t \to T$. We actually have a control with similar-looking but different flavour. Take $S \in [0,T)$ and we have the following equation

$$\frac{\partial}{\partial t}\left((e^{t-S}-1)\frac{\partial u}{\partial t} - u\right) = \Delta\left((e^{t-S}-1)\frac{\partial u}{\partial t} - u\right) + n - \langle \widetilde{\omega}_t, \omega_S \rangle.$$

Proper choice would make sure $\omega_S > 0$ and so $(e^{t-S}-1)\frac{\partial u}{\partial t} - u \leqslant C(S)$. For $t \in (S,T)$, we have

$$\frac{\partial u}{\partial t} \leqslant \frac{C}{T-S} \cdot u + C(S).$$

Even if $u \to -\infty$ as $t \to T$, these two inequalities above say that $\frac{\partial u}{\partial t} \to -\infty$ much faster than u does.

3.2 Strategy for $T < \infty$ case

In this case, we expect that the case of $u \to -\infty$ uniformly as $t \to T$ is not possible. That case means $\max_{X \times \{t\}} u \to -\infty$ as $t \to T$. Since $u(\cdot,t) \in PSH_{\omega_t}(X) \subset PSH_{C\omega_0}(X)$ for some large C and any $t \in [0,T)$, the standard Green's function argument tells us that it is equivalent to $\int_X u(\cdot,t)\Omega \to -\infty$ as $t \to T$. So a reasonable game plan is to search for the uniform lower bound for either $\max_{X\times\{t\}} u$ or $\int_X u(\cdot,t)\Omega$ for $t \in [0,T)$. We only need to consider the volume collapsing case. So $\int_X e^{\frac{\partial u}{\partial t}+u}\Omega = [\omega_t]^n \sim C(T-t)^k$ for some $k \in \{1,\cdots,n\}$. Hence one always has for any $t \in [0,T)$,

$$\max_{X\times\{t\}}\left(\frac{\partial u}{\partial t} + u\right) \geqslant C\log(T-t) - C, \quad \min_{X\times\{t\}}\left(\frac{\partial u}{\partial t} + u\right) \leqslant C\log(T-t) - C.$$

In the following, we list two different additional assumptions to make it work.

- $\operatorname{osc}_{X\times\{t\}}\left(\frac{\partial u}{\partial t} + u\right) \leqslant C - C\log(T-t)$

  Assuming this, we have

  $$\frac{\partial u}{\partial t} + u \geqslant C\log(T-t) - C,$$

  and so $u \geqslant -C$, which is more than what's needed.

  This additional assumption indeed says $\widetilde{\omega}_t^n \geqslant (T-t)^C\Omega$.

- $\frac{\partial u}{\partial t} - \frac{\int_X \frac{\partial u}{\partial t}\Omega}{\int_X \Omega} \leqslant C - C\log(T-t)$

  Since $u \in PSH_{C\omega_0}(X)$, the Green's function argument mentioned earlier gives
  $$u - \frac{\int_X u\Omega}{\int_X \Omega} \leqslant C \text{ uniformly for any } t \in [0,T).$$
  Then we have the following estimation
  $$\begin{aligned} C(T-t)^k &\leqslant \int_X e^{\frac{\partial u}{\partial t}+u}\Omega \\ &= \int_X e^{\frac{\partial u}{\partial t}+u-\frac{\int_X(\frac{\partial u}{\partial t}+u)\Omega}{\Omega}}\Omega \cdot e^{\frac{\int_X(\frac{\partial u}{\partial t}+u)\Omega}{\Omega}} \\ &\leqslant C(T-t)^{-C}\cdot e^{\frac{\int_X(\frac{\partial u}{\partial t}+u)\Omega}{\Omega}}, \end{aligned}$$
  which gives
  $$\int_X \left(\frac{\partial u}{\partial t}+u\right)\Omega \geqslant C\log(T-t) - C.$$
  It's then easy to see $\int_X u \geqslant -C$.

  Notice that this assumption is more or less in the opposite direction of **Case 1** in Subsection 2.3.

3.3 Difference between $T<\infty$ and $T=\infty$ cases

The following example says for the infinite time collapsing case, i.e. $T=\infty$ and $[\omega_T]^n = 0$, it is possible that $u \to -\infty$ uniformly as $t \to T$.

Example 3.1. *Suppose $K_X = [\omega_\infty]$ gives a fibration structure of X with general fibre dimension $0<k\leqslant n$, i.e., $P: X \to \mathbb{CP}^N$ with $mK_X = P^*[\omega_{FS}]$ and $P(X)$ of dimension $n-k$. Then $u \sim -kt$. This can be seen as follows. Begin with the following scalar potential flow*
$$\frac{\partial v}{\partial t} = \log\frac{(\omega_t+\sqrt{-1}\partial\bar\partial v)^n}{\Omega} - v + kt, \qquad v(\cdot,0)=0.$$
Clearly, it still coresspond to the same metric flow (1.1) and the relation between u and v is
$$u = v + f(t) \quad with \quad \frac{df}{dt} + f = -kt, \quad f(0)=0.$$
It's easy to get $f(t)\sim -kt$ and $\frac{df}{dt} \sim -k$. Rewrite the equation of v as follows
$$(\omega_t+\sqrt{-1}\partial\bar\partial v)^n = e^{-kt}e^{\frac{\partial v}{\partial t}+v}\Omega$$
and apply the L^∞ estimates in [2] and [1], we have $|v|\leqslant C$ for all time. Hence $u\sim -kt$ which tends to $-\infty$ as $t\to\infty$.

So the $T = \infty$ collapsing case needs to be treated differently. Intuitively, the difference can be understood as follows. For finite time collapsing case, one can do similar things as in the above example and have the flow for v,

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}v)^n = (T-t)^k e^{\frac{\partial v}{\partial t}+v}\Omega,$$

which corresponds to the same metric flow and

$$u = v + f(t) \;\text{ with }\; \frac{df}{dt} + f = k\log(T-t), \quad f(0) = 0.$$

Now we have $|f| \leqslant C$ and $\frac{df}{dt} \sim k\log(T-t)$. In principle, we expect that v is bounded or at least doesn't tend to $-\infty$ uniformly, and so that's also expected for u.

4 Further Remarks

We now state the following (bold) conjecture. The discussion in this note provides some evidence.

Conjecture 4.1. *For the flow (1.1) with singularity at $T < \infty$, $u \geqslant -C$ for $t \in [0, T)$.*

There is also this basic question regarding singularities of Kähler-Ricci flow: *are they always along analytic varieties?* A little discussion with Professor Frédéric Campana brought this to my attention.

In the global volume collapsing case, this is of course the case because the singularity should be developed everywhere on X. So the non-collpasing case is the interesting one, where the weak flow limit is available for both finite and infinite time singularities case from this work.

Naturally, people would want to check the set $\{u_T = -\infty\}$ which is pluripolar. But it can not be the right choice in sight of the known cases summarized here. In fact, by the discussion in Subsection 2.2, we should probably look at the set $\{x \in X \mid \frac{\partial u}{\partial t} \to -\infty\}$, at least for the finite time singularity. We know that $\frac{\partial u}{\partial t} \leqslant Cu + C$ from $\frac{\partial u}{\partial t} \leqslant \frac{u+nt}{e^t-1}$, and so

$$C\frac{\partial u}{\partial t} - C \leqslant \frac{\partial u}{\partial t} + u \leqslant \frac{\partial u}{\partial t} + C.$$

Using the essential decreasing, we can define the limit of $\frac{\partial u}{\partial t} + u$, V over X. Then $\{x \in X \mid \frac{\partial u}{\partial t} \to -\infty\} = \{x \in X \mid \frac{\partial u}{\partial t} + u \to -\infty\} = \{x \in X \mid V = -\infty\}$, and this set is equal to

$$\cap_{A=1}^{\infty} \cup_{s\in[0,T)} \{x \in X \mid \frac{\partial u}{\partial t} + u + Ce^{-t} \leqslant -A \text{ at } (x,s)\}.$$

The decreasing of $\frac{\partial u}{\partial t} + u + Ce^{-t}$ tells us $\{x \in X \mid \frac{\partial u}{\partial t} + u + Ce^{-t} \leqslant -A \text{ at } (x,s)\}$ is increasing as $s \to T$. Also, the result in [8] implies

$$\cup_{s\in[0,T)}\{x \in X \mid \frac{\partial u}{\partial t} + u + Ce^{-t} \leqslant -A \text{ at } (x,s)\} \neq \varnothing$$

for any A. But we do not know whether $\{x \in X \mid \frac{\partial u}{\partial t} \to -\infty\}$ is always non-empty. Of course, the function V can not have a uniform lower bound, but a priori, it might not take the value $-\infty$. Nevertheless, V is upper semi-continuous, and we can find the lower semi-continuization of it, V_*. Probably, $\{V_* = -\infty\}$ is what one should really be looking at.

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